8.1 – General Linear Transformations

Definition: (analogous to Theorem 1.8.2)
If $T : V \to W$ is a mapping from a vector space V to a vector space W, then T is called a **linear transformation** from V to W if the following two properties hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V and for all scalars k:
a) $T(k\mathbf{u}) = kT(\mathbf{u})$ (homogeneity property)
b) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity property)

In the special case where $V = W$, the linear transformation T is called a **linear operator** on the vector space V.

Theorem 8.1.1 (analogous to Theorem 1.8.1)
If $T : V \to W$ is a linear transformation, then:
a) $T(\mathbf{0}) = \mathbf{0}$.
b) $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in V.
c) $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v}$ in V.

Examples of Linear transformations

- $T : V \to W$ defined by $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$ (the zero transformation)
- $I : V \to V$ defined by $I(\mathbf{v}) = \mathbf{v}$ (the identity operator)
- $T : V \to V$ defined by $T(\mathbf{x}) = c\mathbf{x}$ (contraction if $0 < c < 1$ and dilation if $c > 1$)
- $T : V \to R$ defined by $T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}_0 \rangle$ (inner product of $\mathbf{x}$ with $\mathbf{x}_0$) [We'll see this in Ch. 6]
- $T : M_{nn} \to M_{nn}$ defined by $T(A) = A^T$

- $T : V \to R^n$ defined by $T(f) = (f(x_1), f(x_2), \dots, f(x_n))$, where V is a subspace of $F(-\infty, \infty)$, and $x_1, x_2, \dots, x_n$ is a sequence of real numbers (evaluation transformation)
- $D : C^1(-\infty, \infty) \to F(-\infty, \infty)$ defined by $D(\mathbf{f}) = f'(x)$ (differentiation)
- $J : C(-\infty, \infty) \to C^1(-\infty, \infty)$ defined by $J(\mathbf{f}) = \int_0^x f(t)\, dt$ (integration)

Theorem 8.1.2 Let $T : V \to W$ be a linear transformation, for which the vector space V is finite-dimensional. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V, then the image of any vector $\mathbf{v}$ in V can be expressed as
$T(\mathbf{v}) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \dots + c_n T(\mathbf{v}_n)$ where $c_1, c_2, \dots, c_n$ are the coefficients required to express $\mathbf{v}$ as a linear combination of the vectors in the basis S.

#20 Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{v}_1 = (-2, 1)$ and $\mathbf{v}_2 = (1, 3)$, and let $T : R^2 \to R^3$ be the linear transformation such that $T(\mathbf{v}_1) = (-1, 2, 0)$ and $T(\mathbf{v}_2) = (0, -3, 5)$. Find a formula for $T(x_1, x_2)$ and use that formula to find $T(2, -3)$.

Definition: (analogous to definitions seen in Section 4.2 (**kernel**) and Section 1.8 (**range**); also related to Definition 2 of Section 4.8 (**null space**) and, by Theorem 4.8.1, column space)
If $T : V \to W$ is a linear transformation, then the set of vectors in V that T maps into $\mathbf{0}$ is called the **kernel** of T and is denoted by $\ker(T)$. The set of all vectors in W that are images under T of at least one vector in V is called the **range** of T and is denoted by $R(T)$.

#6 Determine whether the mapping T is a linear transformation, and if so, find its kernel.
$T : M_{22} \to R$, where

a. $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 3a - 4b + c - d$

b. $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a^2 + b^2$

#10 Let $T : P_2 \to P_3$ be the linear transformation defined by $T(p(x)) = xp(x)$. Which of the following are in $\ker(T)$?

a. x^2

b. 0

c. $1 + x$

d. $-x$

#11 Let $T : P_2 \to P_3$ be the linear transformation in Exercise 10. Which of the following are in $R(T)$?

a. $x + x^2$

b. $1 + x$

c. $3 - x^2$

d. $-x$

#25 Let $T_A : R^4 \to R^3$ be multiplication by A. Find a basis for the kernel of T_A, and then find a basis for the range of T_A that consists of column vectors of A.

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{bmatrix}$$

Theorem 8.1.3 If $T : V \to W$ is a linear transformation, then:
a) The kernel of T is a subspace of V.
b) The range of T is a subspace of W.

Definition: (analogous to Definition 1 of Section 4.9)
Let $T : V \to W$ be a linear transformation. In the case that the range of T is finite-dimensional its dimension is called the **rank of T**, denoted by rank(T); and if the kernel of T is finite-dimensional, then its dimension is called the **nullity of T**, denoted by nullity(T).

Theorem 8.1.4 Dimension Theorem for Linear Transformations
(generalization of Theorem 4.9.2)
If $T : V \to W$ is a linear transformation from a finite-dimensional vector space V to a vector space W, then the range of T is finite-dimensional, and
$\text{rank}(T) + \text{nullity}(T) = \dim(V)$.

#13 In each part, use the given information to find the nullity of the linear transformation T.
a. $T : R^5 \to P_5$ has rank 3.
b. $T : P_4 \to P_3$ has rank 1.
c. The range of $T : M_{mn} \to R^3$ is R^3.
d. $T : M_{22} \to M_{22}$ has rank 3.